1. *(a) Prove that pivot columns in an upper echelon matrix are linearly independent*

   Solution: Let A be an $m \times n$ matrix which is in upper echelon form and without loss of generality let v_1, v_2,...,v_k, $k \leq n$ be the pivot columns of A. Also let $v_j = (v_{1j}, v_{2j}, ..., v_{mj})^t$, $1 \leq j \leq k$. Now let $\sum_{j=1}^{k} c_j v_j = 0$. Then $c_k = 0$ because if v_{lk} is the last non-zero entry of v_j, $v_{lj} = 0$ for all $j \leq k$ since the matrix A is in upper echelon form. Similar arguements also show that $c_j = 0$ for $1 \leq j \leq (k-1)$ as well. Hence pivot columns are linearly independent. $\square$

   (b) Let A be a $p \times q$ upper echelon matrix with k pivots. If $k \leq p < q$, prove that A is not one-one.

   Solution: A is a linear transformation from $\mathbb{R}^q$ to $\mathbb{R}^p$ and the range of A is the linear span of the pivot columns. Therefore range of A has dimension $k < q$. Then the rank nulity theorem tells us that the null space of A has positive dimension. Hence A is not one-one. $\square$

2. *(a) Find the QR decomposition of* $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

   Solution: Let $u_1 = (1,1,0,1)^t$, $u_2 = (1,2,0,1)^t$, $u_3 = (2,0,1,0)^t$. Let $v_1 = u_1$, $v_2 = u_2 - \frac{<u_2,u_1>}{||u_1^2||}$ and $v_3 = \frac{<u_3,u_1>}{||u_1||^2}u_1 - \frac{<u_3,u_2>}{||u_2||^2}u_2$. Let $e_i = \frac{v_i}{||v_i||}$ for $1 \leq i \leq 3$. Let Q be the matrix consists of e_1, e_2, e_3 as its columns vectors. Then
   Q is an orthogonal matrix since e_i and e_j are orthogonal to each other if $i \neq j$ and $||e_i|| = 1$, for $i, j \in \{1,2,3\}$. Also let $R = \begin{pmatrix} <e_1,u_1> & <e_1,u_2> & <e_1,u_3> \\ 0 & <e_2,u_2> & <e_2,u_3> \\ 0 & 0 & <e_3,u_3> \end{pmatrix}$.

   Then $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = QR = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{3}{\sqrt{43}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{4}{\sqrt{43}} \\ 0 & 0 & \frac{3}{\sqrt{43}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{3}{\sqrt{43}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & \frac{4}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 0 & \frac{9}{\sqrt{43}} \end{pmatrix}$ is the desired decomposition.

   (b) Give a 3×3 matrix to show that QR decomposition is not unique.

   Solution: Consider the matrix $A = \begin{pmatrix} 1 & -2 & -3 \\ 1 & 4 & 7 \\ 0 & 0 & 1 \end{pmatrix}$. Then

   $$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} & 2\sqrt{2} \\ 0 & 3\sqrt{2} & 5\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \\ 0 & -3\sqrt{2} & -5\sqrt{2} \\ 0 & 0 & -1 \end{pmatrix}$$

   are two QR decompositions of A.

3. *(a) Prove that every $p \times q$ matrix has singular value decomposition.*

Solution: Let A be a $p \times q$ complex matrix. Then A^*A is a $q \times q$ Hermitian matrix and therefore there exists an orthonormal basis $\{u_1, u_2, ..., u_q\}$ of $\mathbb{C}^q$ consisting of eigenvectors of A^*A with corresponding eigenvalues $s_1 \geq s_2 \geq ... \geq s_q \geq 0$.

Let $\sigma_i > 0$ be such that $\sigma_i^2 = s_i$ for $1 \leq i \leq q$. Let $U = [u_1, ..., u_q]$ that is U is the matrix consists of u_i's as its column vectors. Then U is a unitary matrix and $U^*A^*AU = D_q = \text{diag}(s_1, ..., s_q)$, $\text{diag}(s_1, ..., s_q)$ denoting the diagonal matrix with diagonal entries $s_1, ..., s_q$. Now $||A(u_i)||^2 =< A(u_i), A(u_i) >=< A^*A(u_i), u_i >= s_i = \sigma_i^2$. Let r be the largest integer such that $s_r > 0$ (note that if $r = 0$, then A is the zero matrix).

let $v_i = \frac{1}{\sigma_i} A(u_i)$, $1 \leq i \leq r$. Then $< v_i, v_j >= \frac{1}{\sigma_i \sigma_j} < A^*A(u_i), u_j >= \delta_{ij}$. Which shows that v_i's are orthonormal. If $r < p$, then we choose $v_{r+1}, ..., v_p$ such that $\{v_1, ..., v_p\}$ is an orthonormal basis of $\mathbb{C}^p$. Now let $V = [v_1, ..., v_p]$, then V is a unitary matrix. Also let $D = \text{diag}(\sigma_1, ..., \sigma_r)$. Then $AU = VD$ and therefore $A = VDU^*$, which is the singular value decomposition of A.

(b) Prove that the singular values of A and A^* coincide

Solution. If $A = VDU^*$ is the singular value decomposition of A, then taking conjugates on both sides we get the singular value decomposition of A^* as $A^* = UDV^*$. Which shows that the singular values of A and A^* coincide.

4. *(a) Let $A \in M_{p \times q}(\mathbb{C})$ and s_1 be the first singular value. Prove that $s_1 = ||A||$.*

Solution: Let $A = VDU^*$ be the singular value decomposition of A with singular values $s_1 \geq s_2 \geq ... \geq s_q$. Let $x \in \mathbb{C}^q$ and $y = U^*(x)$. Now

$$||Ax||^2 = ||VDU^*x||^2 = ||DU^*x||^2 (\text{as } V \text{ is unitary}) = ||Dy||^2 = \sum_{i=1}^{q} s_i^2 y_i^2 \leq s_1^2 ||y||^2 = s_1^2 ||x||^2, \quad (1)$$

since U is unitary. Therefore, $||A|| \leq s_1$.

Now let u_1 be the first column of U. Then $||u_1|| = 1$ and

$$||Au_1|| = ||DU^*u_1|| = De_1 = s_1,$$

where $e_1 = (1, 0, ..., 0)^t$. This shows that $||A|| = s_1$.

(b) If rank of a $p \times q$ matrix A is q, prove that $s_q = min_{||x||=1} ||Ax||$

Solution: A similar calculation as in (1) above gives that $||Ax|| \geq s_q$. On the other hand if u_q is the qth column of U, then $||u_q|| = 1$ and a similar calculation as above shows that $||Au_q|| = s_q$. Hence we get the desired result.

5. *Let S be a subspace of $\mathbb{R}^n$ and $a, v \in \mathbb{R}^n$. Denote $W = a + S = \{a + x | x \in S\}$.*
(a) Prove that v can be written uniquely as $w + y$ for $w \in W$ and $y \in S^\perp$.

Solution: Let the dimension of S be k and let $\{u_1, u_2, ..., u_k\}$ be a basis of S and let $\{u_{k+1}, ..., u_n\}$ be a basis of $S^\perp$ such that $\{u_1, ..., u_n\}$ is an orthonormal basis of $\mathbb{R}^n$. Then every v can be uniquely written as

$$v = c_1 u_1 + ... + c_k u_k + c_{k+1} u_{k+1} + ... + c_n u_n, c_i \in \mathbb{R}. \quad (2)$$

Now if $a = \sum_{i=1}^{n} a_i u_i$, then

$$v = c_1u_1 + ... + c_ku_k + c_{k+1}u_{k+1} + ... + c_nu_n = \sum_{i=1}^{k} c_iu_i + \sum_{i=k+1}^{n} a_iu_i + \sum_{i=k+1}^{n} (c_i - a_i)u_i. \tag{3}$$

Note that $\sum_{i=1}^{k} c_iu_i + \sum_{i=k+1}^{n} a_iu_i \in W$ and $\sum_{i=k+1}^{n} (c_i - a_i)u_i \in S^{\perp}$. As the expression in (2) is unique, it follows that the expression in (3) is also unique.

(b) Prove that $min_{y \in W}||v - y||$ has a unique solution.

Solution: Let $v = v_W + v_{S^\perp}$ with $v_W \in W$ and $v_{S^\perp} \in S^\perp$. For any $y \in W$,

$$||v - y||^2 = ||(v_w - y) + v_{S^\perp}||^2 = ||v_w - y||^2 + ||v_{S^\perp}||^2 \geq ||v_{S^\perp}||^2 \tag{4}$$

and if $y = v_W$, then $||v - y||^2 = ||v_{S^\perp}||^2$. So $\min_{y \in W}||v - y|| = ||v - v_W||$.

To show the uniqueness, let $y \in W$ be such that $||v - y||^2 = ||v_{S^\perp}||^2$. Then writing $v = v_W + v_{S^\perp}$ we get $||v_w - y||^2 + ||v_{S^\perp}||^2 = ||v_{S^\perp}||^2$, which implies that $y = v_W$.

6. *(a) If A is a non-negative and irreducible $n \times n$ matrix, prove that $(I + A)^{n-1}$ is positive.*

Solution: $(I+A)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} A^k$. Now suppose for some $i, j \in \{1, ..., n\}$, $\left\langle (I + A)^{n-1}e_i, e_j \right\rangle = 0$. Then $\sum_{k=0}^{n-1} \binom{n-1}{k} \left\langle A^k e_i, e_j \right\rangle = 0$, which implies that $\left\langle A^k e_i, e_j \right\rangle = 0$ for all $0 \leq k \leq n - 1$, contradicting the irreducibilty of A. Hence $(I + A)^{n-1}$ is a positive matrix.

(b) Let T and S be non-negative matrices such that T is irreducible and $T - S$ is non-negative. Prove that $spr(T) \geq spr(S)$ and the equality occurs only if $T = S$.

Solution: Let $\beta \in spec(S)$ be such that $|\beta| = spr(S)$. Let $y = (y_1, ..., y_n)^t \in \mathbb{C}^n$ be such that $Sy = \beta y$. Let $v = (v_1, ..., v_n)^t = (|y_1|, ..., |y_n|)^t$, then

$$|\beta|v_i = |\beta y_i| = \left| \sum_{j=1}^{n} S_{ij}y_j \right| \leq \sum_{j=1}^{n} S_{ij}v_j \leq \sum_{j=1}^{n} T_{ij}v_j, \tag{5}$$

which implies that

$$spr(S) = |\beta| \leq \delta_T(v) \leq spr(T),$$

where $\delta_T(v) = min_{1 \leq i \leq n} \left\{ \frac{<Tv, e_i>}{<v, e_i>} :< v, e_i >> 0 \right\}$, e_i denoting the ith member of the standard basis of $\mathbb{R}^n$.

Now let $spr(S) = spr(T)$, we will show that $S = T$. From (5) we have $T(v) - spr(S)v \geq 0$ and therefore, $T(v) - spr(T)v \geq 0$ since $spr(S) = spr(T)$. If $T(v) - spr(T)v > 0$, then

$$< Tv, e_i >> spr(T) < v, e_i > .$$

This implies that $\delta_T(v) > spr(T)$, which is a contradiction. Hence $T(v) = spr(T)v$. Since T is also irreducible, it follows that $v_i > 0$ for $1 \leq i \leq n$.

Since $\beta v = Tv$ and $v_i > 0$, we get

$$\sum_{j=1}^{n} T_{ij} v_j = spr(T) v_i = spr(S) v_i = \sum_{j=1}^{n} S_{ij} v_j,$$

which implies that $\sum_{j=1}^{n} (T_{ij} - S_{ij}) v_j = 0$. Hence $S = T$.

7. *(a) Let A be a non-negative irreducible matrix. If λ is an eigenvalue of A with eigenvector $x \geq 0$, prove that λ is the spectral radius of A.*

Solution: First we show that actually $x > 0$. If not, let $x_i = 0$ for some $i \in \{1, ..., n\}$ (assuming A to be an $n \times n$ matrix). Then for any $k \in \mathbb{N}$, $\sum_{j=0}^{n} A_{ij}^k x_j = \lambda^k x_i = 0$. Since x can not be a zero vector, it follows that $A_{ij}^k = 0$ for some j and for any $k \in \mathbb{N}$ contradicting the irreducibility of A.

Now let μ be the spectral radius (which is also an eigenvalue in this case) of A and let $u > 0$ be such that $Au = \mu u$. If we choose $\varepsilon > 0$ very small then $u - \varepsilon v > 0$ as well. By the property of spectral radius we know that there is some $i \in \{1, ..., n\}$ such that

$$\frac{\langle A(u - \varepsilon x), e_i \rangle}{\langle u - \varepsilon x, e_i \rangle} \leq \mu.$$

From which it follows after a simple calculation that $\mu \leq \lambda$. Hence λ is the spectral radius of A.

(b) Prove that $A = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$ is irreducible and find its Perron-pair.

Solution: We have $A^2 = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 9 & 0 \\ 6 & 0 & 6 \end{pmatrix}$. Therefore comparing A and A^2 we see that if the ijth entry of A is 0, then the ijth entry of A^2 is positive. Therefore, A is irreducible.

The characteristic equation for A is $x^3 - 9x = 0$, therefore, the Frobenius eigenvalue is 3. A routine calculation shows that the vector $v = (\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}})^t$ is an eigenvector corresponding to the eigenvalue 3 with $||v|| = 1$. Hence the Perron-pair of the matrix A is $(3, (\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}})^t)$.